

Direct and inverse theorems for Bernstein operators with inner singularities

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Abstract

We introduce a new type of Bernstein operators, which can be used to approximate the functions with inner singularities. The direct and inverse results of the weighted approximation of this new type of combinations are obtained.

Keywords: weighted approximation; Bernstein operators; inner singularities

1 Introduction

The set of all continuous functions, defined on the interval I , is denoted by $C(I)$. For any $f \in C([0, 1])$, the corresponding Bernstein operators are defined as follows:

$$B_n(f, x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, 2, \dots, n, \quad x \in [0, 1].$$

Recently Felten showed the following two theorems in [1]:

Theorem A. Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0, 1] \rightarrow R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness([4]) such that ϕ^2 and φ^2/ϕ^2 are concave. Then, for $f \in C[0, 1]$ and $0 < \alpha < 2$, $|B_n(f, x) - f(x)| \leq \omega_\phi^2(f, n^{-1/2} \frac{\varphi(x)}{\phi(x)})$.

Theorem B. Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0, 1] \rightarrow R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness such that ϕ^2 and φ^2/ϕ^2 are concave. Then, for $f \in C[0, 1]$ and $0 < \alpha < 2$, $|B_n(f, x) - f(x)| = O((n^{-1/2} \frac{\varphi(x)}{\phi(x)})^\alpha)$ implies $\omega_\phi^2(f, t) = O(t^\alpha)$.

Approximation properties of Bernstein operators have been studied very well (see [2], [3], [4]-[7], for example). In order to approximate the functions with singularities, Della Vecchia

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et al. [3] introduced some kinds of modified Bernstein operators. Throughout the paper, C denotes a positive constant independent of n and x , which may be different in different cases. Let $\phi : [0, 1] \rightarrow R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness, that is, ϕ satisfies the following conditions:

- (I) For every proper subinterval $[a, b] \subseteq [0, 1]$ there exists a constant $C_1 \equiv C(a, b) > 0$ such that $C_1^{-1} \leq \phi(x) \leq C_1$ for $x \in [a, b]$.
(II) There are two numbers $\beta(0) \geq 0$ and $\beta(1) \geq 0$ for which

$$\phi(x) \sim \begin{cases} x^{\beta(0)}, & \text{as } x \rightarrow 0+, \\ (1-x)^{\beta(1)}, & \text{as } x \rightarrow 1-. \end{cases}$$

($X \sim Y$ means $C^{-1}Y \leq X \leq CY$ for some C).

Combining conditions (I) and (II) on ϕ , we can deduce that

$$C^{-1}\phi_2(x) \leq \phi(x) \leq C\phi_2(x), \quad x \in [0, 1],$$

where $\phi_2(x) = x^{\beta(0)}(1-x)^{\beta(1)}$.

2 The Main Results

Let $\bar{w}(x) = |x - \xi|^\alpha$, $0 < \xi < 1$, $\alpha > 0$ and $C_{\bar{w}} := \{f \in C([0, 1] \setminus \{\xi\}) : \lim_{x \rightarrow \xi} (\bar{w}f)(x) = 0\}$.

The norm in $C_{\bar{w}}$ is defined by $\|f\|_{C_{\bar{w}}} := \|\bar{w}f\| = \sup_{0 \leq x \leq 1} |(\bar{w}f)(x)|$. Define

$$W_\phi^2 := \{f \in C_{\bar{w}} : f' \in A.C.((0, 1)), \|\bar{w}\phi^2 f''\| < \infty\},$$

$$W_{\bar{w}, \lambda}^2 := \{f \in C_{\bar{w}} : f' \in A.C.((0, 1)), \|\bar{w}\varphi^{2\lambda} f''\| < \infty\}.$$

For $f \in C_{\bar{w}}$, the weighted modulus of smoothness is defined by

$$\omega_\phi^2(f, t)_{\bar{w}} := \sup_{0 < h \leq t} \sup_{0 \leq x \leq 1} |\bar{w}(x) \Delta_{h\phi(x)}^2 f(x)|,$$

where

$$\Delta_{h\phi}^2 f(x) = f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x)),$$

and $\varphi(x) = \sqrt{x(1-x)}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$.

Let

$$\psi(x) = \begin{cases} 10x^3 - 15x^4 + 6x^5, & 0 < x < 1, \\ 0, & x \leq 0, \\ 1, & x \geq 1. \end{cases}$$

Obviously, ψ is non-decreasing on the real axis, $\psi \in C^2((-\infty, +\infty))$, $\psi^{(i)}(0) = 0$, $i = 0, 1, 2$. $\psi^{(i)}(1) = 0$, $i = 1, 2$ and $\psi(1) = 1$. Further, let

$$x_1 = \frac{[n\xi - 2\sqrt{n}]}{n}, \quad x_2 = \frac{[n\xi - \sqrt{n}]}{n}, \quad x_3 = \frac{[n\xi + \sqrt{n}]}{n}, \quad x_4 = \frac{[n\xi + 2\sqrt{n}]}{n},$$

and

$$\bar{\psi}_1(x) = \psi\left(\frac{x - x_1}{x_2 - x_1}\right), \quad \bar{\psi}_2(x) = \psi\left(\frac{x - x_3}{x_4 - x_3}\right).$$

Consider

$$P(x) := \frac{x - x_4}{x_1 - x_4} f(x_1) + \frac{x_1 - x}{x_1 - x_4} f(x_4),$$

the linear function joining the points $(x_1, f(x_1))$ and $(x_4, f(x_4))$. And let

$$\bar{F}_n(f, x) := \bar{F}_n(x) = f(x)(1 - \bar{\psi}_1(x) + \bar{\psi}_2(x)) + \bar{\psi}_1(x)(1 - \bar{\psi}_2(x))P(x).$$

From the above definitions it follows that

$$\bar{F}_n(f, x) = \begin{cases} f(x), & x \in [0, x_1] \cup [x_4, 1], \\ f(x)(1 - \bar{\psi}_1(x)) + \bar{\psi}_1(x)P(x), & x \in [x_1, x_2], \\ P(x), & x \in [x_2, x_3], \\ P(x)(1 - \bar{\psi}_2(x)) + \bar{\psi}_2(x)f(x), & x \in [x_3, x_4]. \end{cases}$$

Evidently, \bar{F}_n is a positive linear operator which depends on the functions values $f(k/n)$, $0 \leq k/n \leq x_2$ or $x_3 \leq k/n \leq 1$, it reproduces linear functions, and $\bar{F}_n \in C^2([0, 1])$ provided $f \in W_\phi^2$. Now for every $f \in C_{\bar{w}}$ define the Bernstein type operator

$$\begin{aligned} \bar{B}_n(f, x) &:= B_n(\bar{F}_n(f), x) \\ &= \sum_{k/n \in [0, x_1] \cup [x_4, 1]} p_{n,k}(x) f\left(\frac{k}{n}\right) + \sum_{x_2 < k/n < x_3} p_{n,k}(x) P\left(\frac{k}{n}\right) \\ &\quad + \sum_{x_1 < k/n < x_2} p_{n,k}(x) \left\{ f\left(\frac{k}{n}\right) (1 - \bar{\psi}_1\left(\frac{k}{n}\right)) + \bar{\psi}_1\left(\frac{k}{n}\right) P\left(\frac{k}{n}\right) \right\} \\ &\quad + \sum_{x_3 < k/n < x_4} p_{n,k}(x) \left\{ P\left(\frac{k}{n}\right) (1 - \bar{\psi}_2\left(\frac{k}{n}\right)) + \bar{\psi}_2\left(\frac{k}{n}\right) f\left(\frac{k}{n}\right) \right\}. \end{aligned} \quad (2.1)$$

Obviously, \bar{B}_n is a positive linear operator, $\bar{B}_n(f)$ is a polynomial of degree at most n , it preserves linear functions, and depends only on the function values $f(k/n)$, $k/n \in [0, x_2] \cup [x_3, 1]$. Now we state our main results as follows:

Theorem 1. *If $\alpha > 0$, for any $f \in C_{\bar{w}}$, we have*

$$\|\bar{w} \bar{B}_n''(f)\| \leq C n^2 \|\bar{w} f\|. \quad (2.2)$$

Theorem 2. *For any $\alpha > 0$, $\min\{\beta(0), \beta(1)\} \geq \frac{1}{2}$, $0 < \xi < 1$, we have*

$$|\bar{w}(x) \phi^2(x) \bar{B}_n''(f, x)| \leq \begin{cases} C n \|\bar{w} f\|, & f \in C_{\bar{w}}, \\ C \|\bar{w} \phi^2 f''\|, & f \in W_\phi^2. \end{cases} \quad (2.3)$$

Theorem 3. *For $f \in C_{\bar{w}}$, $0 < \xi < 1$, $\alpha > 0$, $\min\{\beta(0), \beta(1)\} \geq \frac{1}{2}$, $\alpha_0 \in (0, 2)$, we have*

$$\bar{w}(x) |f(x) - \bar{B}_n(f, x)| = O((n^{-\frac{1}{2}} \phi^{-1}(x) \delta_n(x))^{\alpha_0}) \iff \omega_\phi^2(f, t)_{\bar{w}} = O(t^{\alpha_0}).$$

3 Lemmas

Lemma 1. ([10]) *For any non-negative real u and v , we have*

$$\sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^{-u} \left(1 - \frac{k}{n}\right)^{-v} p_{n,k}(x) \leq C x^{-u} (1-x)^{-v}. \quad (3.1)$$

Lemma 2.([3]) For any $\alpha > 0$, $f \in C_{\bar{w}}$, we have

$$\|\bar{w}\bar{B}_n(f)\| \leq C\|\bar{w}f\|. \quad (3.2)$$

Lemma 3.([9]) Let $\min\{\beta(0), \beta(1)\} \geq \frac{1}{2}$, then for $0 < t < \frac{1}{4}$ and $t < x < 1 - t$, we have

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \phi^{-2}(x + \sum_{k=1}^2 u_k) du_1 du_2 \leq Ct^2 \phi^{-2}(x). \quad (3.3)$$

Proof. From the definition of $\phi(x)$, it is enough to prove (3.3) for $t < x \leq \frac{1}{2}$ since the proof for $\frac{1}{2} < x < 1 - t$ is very similar. Obviously, we have

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{1}{(x + \sum_{k=1}^2 u_k)} du_1 du_2 \leq Ct^2 x^{-1}.$$

Therefore, by the Hölder inequality, we have

$$\begin{aligned} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \phi^{-2}(x + \sum_{k=1}^2 u_k) du_1 du_2 &\leq C\left(\frac{16}{7}\right)^{2\beta(1)} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{1}{(x + \sum_{k=1}^2 u_k)^{2\beta(0)}} du_1 du_2 \\ &\leq C\left(\frac{16}{7}\right)^{2\beta(1)} t^{2(1-2\beta(0))} \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{1}{(x + \sum_{k=1}^2 u_k)} du_1 du_2 \right)^{2\beta(0)} \\ &\leq C\left(\frac{16}{7}\right)^{2\beta(1)} t^2 x^{-2\beta(0)}. \square \end{aligned}$$

Lemma 4.([3]) If $\gamma \in R$, then

$$\sum_{k=0}^n p_{n,k}(x) |k - nx|^\gamma \leq Cn^{\frac{\gamma}{2}} \varphi^\gamma(x). \quad (3.4)$$

Lemma 5. Let $A_n(x) := \bar{w}(x) \sum_{|k-n\xi| \leq \sqrt{n}} p_{n,k}(x)$. Then $A_n(x) \leq Cn^{-\frac{\alpha}{2}}$ for $0 < \xi < 1$ and $\alpha > 0$.

Proof. If $|x - \xi| \leq \frac{3}{\sqrt{n}}$, then the statement is trivial. Hence assume $0 \leq x \leq \xi - \frac{3}{\sqrt{n}}$ (the case $\xi + \frac{3}{\sqrt{n}} \leq x \leq 1$ can be treated similarly). Then for a fixed x the maximum of $p_{n,k}(x)$ is attained for $k = k_n := [n\xi - \sqrt{n}]$. By using Stirling's formula, we get

$$\begin{aligned} p_{n,k_n}(x) &\leq C \frac{\left(\frac{n}{e}\right)^n \sqrt{n} x^{k_n} (1-x)^{n-k_n}}{\left(\frac{k_n}{e}\right)^{k_n} \sqrt{k_n} \left(\frac{n-k_n}{e}\right)^{n-k_n} \sqrt{n-k_n}} \\ &\leq \frac{C}{\sqrt{n}} \left(\frac{nx}{k_n}\right)^{k_n} \left(\frac{n(1-x)}{n-k_n}\right)^{n-k_n} \\ &= \frac{C}{\sqrt{n}} \left(1 - \frac{k_n - nx}{k_n}\right)^{k_n} \left(1 + \frac{k_n - nx}{n - k_n}\right)^{n-k_n}. \end{aligned}$$

Now from the inequalities

$$k_n - nx = [n\xi - \sqrt{n}] - nx > n(\xi - x) - \sqrt{n} - 1 \geq \frac{1}{2}n(\xi - x),$$

and

$$1 - u \leq e^{-u - \frac{1}{2}u^2}, \quad 1 + u \leq e^u, \quad u \geq 0,$$

it follows that the second inequality is valid. To prove the first one we consider the function $\lambda(u) = e^{-u-\frac{1}{2}u^2} + u - 1$. Here $\lambda(0) = 0$, $\lambda'(u) = -(1+u)e^{-u-\frac{1}{2}u^2} + 1$, $\lambda'(0) = 0$, $\lambda''(u) = u(u+2)e^{-u-\frac{1}{2}u^2} \geq 0$, whence $\lambda(u) \geq 0$ for $u \geq 0$. Hence

$$\begin{aligned} p_{n,k_n}(x) &\leq \frac{C}{\sqrt{n}} \exp\left\{k_n\left[-\frac{k_n-nx}{k_n} - \frac{1}{2}\left(\frac{k_n-nx}{k_n}\right)^2\right] + k_n - nx\right\} \\ &= \frac{C}{\sqrt{n}} \exp\left\{-\frac{(k_n-nx)^2}{2k_n}\right\} \leq e^{-Cn(\xi-x)^2}. \end{aligned}$$

Thus $A_n(x) \leq C(\xi-x)^\alpha e^{-Cn(\xi-x)^2}$. An easy calculation shows that here the maximum is attained when $\xi-x = \frac{C}{\sqrt{n}}$ and the lemma follows. \square

Lemma 6. For $0 < \xi < 1$, $\alpha, \beta > 0$, we have

$$\bar{w}(x) \sum_{|k-n\xi| \leq \sqrt{n}} |k-nx|^\beta p_{n,k}(x) \leq Cn^{\frac{\beta-\alpha}{2}} \varphi^\beta(x). \quad (3.5)$$

Proof. By (3.4) and the lemma 5, we have

$$\bar{w}(x)^{\frac{1}{2n}} (\bar{w}(x) \sum_{|k-n\xi| \leq \sqrt{n}} p_{n,k}(x))^{\frac{2n-1}{2n}} \left(\sum_{|k-n\xi| \leq \sqrt{n}} |k-nx|^{2n\beta} p_{n,k}(x) \right)^{\frac{1}{2n}} \leq Cn^{\frac{\beta-\alpha}{2}} \varphi^\beta(x). \square$$

Lemma 7. For any $\alpha > 0$, $f \in W_\phi^2$, $\min\{\beta(0), \beta(1)\} \geq \frac{1}{2}$, we have

$$\bar{w}(x) |f(x) - P(f, x)|_{[x_1, x_4]} \leq C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 f''\|. \quad (3.6)$$

Proof. If $x \in [x_1, x_4]$, for any $f \in W_\phi^2$, we have

$$\begin{aligned} f(x_1) &= f(x) + f'(x)(x_1 - x) + \int_{x_1}^x (t - x_1) f''(t) dt, \\ f(x_4) &= f(x) + f'(x)(x_4 - x) + \int_{x_4}^x (t - x_4) f''(t) dt, \\ \delta_n(x) &\sim \frac{1}{\sqrt{n}}, \quad n = 1, 2, \dots \end{aligned}$$

So

$$\begin{aligned} \bar{w}(x) |f(x) - P(f, x)| &\leq \bar{w}(x) \left| \frac{x - x_4}{x_1 - x_4} \int_{x_1}^x |(t - x_1) f''(t)| dt \right. \\ &\quad \left. + \frac{x_1 - x}{x_1 - x_4} \int_{x_4}^x |(t - x_4) f''(t)| dt \right| \\ &:= I_1 + I_2. \end{aligned}$$

Whence t between x_1 and x , we have $\frac{|t-x_1|}{\bar{w}(t)} \leq \frac{|x-x_1|}{\bar{w}(x)}$, then

$$\begin{aligned} I_1 &\leq Cn^{\frac{1}{2}} \|\bar{w}\phi^2 f''\| |(x-x_1)(x-x_4)| \int_{x_1}^x \phi^{-2}(t) dt \leq C \left(\frac{\varphi(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 f''\| \\ &\leq C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 f''\|. \end{aligned}$$

Analogously, we have

$$I_2 \leq C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 f''\|.$$

Now the lemma follows from combining these results together. \square

Lemma 8. *If $f \in W_\phi^2$, $\min\{\beta(0), \beta(1)\} \geq \frac{1}{2}$, then*

$$\|\bar{w}\phi^2\bar{F}_n''\| = O(\|\bar{w}\phi^2f''\|). \quad (3.7)$$

Proof. Again, it is sufficient to estimate $(\bar{w}\phi^2\bar{F}_n'')(x)$ for $x \in [x_3, x_4]$, and the same as $x \in [x_1, x_2]$. For $x \in [x_2, x_3]$, $\bar{F}_n''(x) = 0$, while for $x \in [0, x_1] \cup [x_4, 1]$, $\bar{F}_n(x) = f(x)$. Thus for $x \in [x_3, x_4]$, then $\bar{F}_n(x) = P(x) + \bar{\psi}_2(x)(f(x) - P(x))$ and

$$\begin{aligned} \bar{F}_n''(x) &= n\psi''[n^{\frac{1}{2}}(x - x_3)](f(x) - P(x)) \\ &\quad + 2n^{\frac{1}{2}}\psi'[n^{\frac{1}{2}}(x - x_3)](f(x) - P(x))' + \psi[n^{\frac{1}{2}}(x - x_3)]f''(x) \\ &:= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

From the proof of lemma 7, we have

$$\begin{aligned} |\bar{w}(x)\phi^2(x)I_1(x)| &= O(n\phi^2(x)\psi''[n^{\frac{1}{2}}(x - x_3)]\bar{w}(x)(f(x) - P(x))) \\ &= O(n\phi^2(x) \cdot (\frac{\varphi(x)}{\sqrt{n}\phi(x)})^2 \|\bar{w}\phi^2f''\|) \\ &= O(\|\bar{w}\phi^2f''\|). \end{aligned}$$

For $I_3(x)$, it is obvious that

$$|\bar{w}(x)\phi^2(x)I_3(x)| = O(\|\bar{w}\phi^2f''\|).$$

Finally

$$\begin{aligned} |\bar{w}(x)\phi^2(x)I_2(x)| &= O(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)|f'(x) - P'(x)|) \\ &= O(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)|f'(x) - n^{\frac{1}{2}}\int_{x_1}^{x_4} f'(t)dt|) \\ &= O(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)|n^{\frac{1}{2}}\int_{x_1}^{x_4}\int_t^x f''(u)dudt|) \\ &= O(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)|\int_{x_1}^{x_4} f''(u)du|) \\ &= O(\|\bar{w}\phi^2f''\|). \square \end{aligned}$$

4 Proof of Theorem

4.1 Proof of Theorem 1

Case 1. If $f \in C_{\bar{w}}$, when $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$, by [2], we have

$$\begin{aligned} |\bar{w}(x)\bar{B}_n''(f, x)| &\leq n\varphi^{-2}(x)\bar{w}(x)|\bar{B}_n(f, x)| \\ &\quad + \bar{w}(x)\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x)|k - nx||\bar{F}_n(\frac{k}{n})| \\ &\quad + \bar{w}(x)\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x)(k - nx)^2|\bar{F}_n(\frac{k}{n})| \\ &:= A_1 + A_2 + A_3. \end{aligned} \quad (4.1)$$

By (3.2), we have

$$A_1(x) = n\varphi^{-2}(x)\bar{w}(x)|\bar{B}_n(f, x)| \leq Cn^2\|\bar{w}f\|. \quad (4.2)$$

and

$$A_2 = \bar{w}(x)\varphi^{-4}(x) \left[\sum_{k/n \in A} |k - nx| |\bar{F}_n(\frac{k}{n})| p_{n,k}(x) + \sum_{x_2 \leq k/n \leq x_3} |k - nx| |P(\frac{k}{n})| p_{n,k}(x) \right] := \sigma_1 + \sigma_2.$$

thereof $A := [0, x_2] \cup [x_3, 1]$. If $\frac{k}{n} \in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n})} \leq C(1+n^{-\frac{\alpha}{2}}|k-nx|^\alpha)$, we have $|k-n\xi| \geq \frac{\sqrt{n}}{2}$, by (3.4), then

$$\begin{aligned} \sigma_1 &\leq C\|\bar{w}f\|\varphi^{-4}(x) \sum_{k=0}^n p_{n,k}(x)|k-nx|[1+n^{-\frac{\alpha}{2}}|k-nx|^\alpha] \\ &= C\|\bar{w}f\|\varphi^{-4}(x) \sum_{k=0}^n p_{n,k}(x)|k-nx| + Cn^{-\frac{\alpha}{2}}\|\bar{w}f\|\varphi^{-4}(x) \sum_{k=0}^n p_{n,k}(x)|k-nx|^{1+\alpha} \\ &\leq Cn^{\frac{1}{2}}\varphi^{-3}(x)\|\bar{w}f\| + Cn^{\frac{1}{2}}\varphi^{-3+\alpha}(x)\|\bar{w}f\| \\ &\leq Cn^2\|\bar{w}f\|. \end{aligned}$$

For σ_2 , P is a linear function. We note $|P(\frac{k}{n})| \leq \max(|P(x_1)|, |P(x_4)|) := P(a)$. If $x \in [x_1, x_4]$, we have $\bar{w}(x) \leq \bar{w}(a)$. So, if $x \in [x_1, x_4]$, by (3.4), then

$$\sigma_2 \leq C\bar{w}(a)P(a)\varphi^{-4}(x) \sum_{k=0}^n p_{n,k}(x)|k-nx| \leq Cn^2\|\bar{w}f\|.$$

If $x \notin [x_1, x_4]$, then $\bar{w}(a) > n^{-\frac{\alpha}{2}}$, by (3.5), we have

$$\begin{aligned} \sigma_2 &\leq C\bar{w}(x)\varphi^{-4}(x) \sum_{x_2 \leq k/n \leq x_3} |P(a)|(k-nx)|p_{n,k}(x) \\ &\leq Cn^{\frac{\alpha}{2}}\|\bar{w}f\|\varphi^{-4}(x)\bar{w}(x) \sum_{x_2 \leq k/n \leq x_3} |k-nx|p_{n,k}(x) \\ &\leq Cn^2\|\bar{w}f\|. \end{aligned}$$

So

$$A_2 \leq Cn^2\|\bar{w}f\|. \quad (4.3)$$

Similarly

$$A_3 \leq Cn^2\|\bar{w}f\|. \quad (4.4)$$

It follows from combining with (4.1)-(4.4) that the inequality is proved.

Case 2. When $x \in [0, \frac{1}{n}]$ (The same as $x \in [1 - \frac{1}{n}, 1]$), by [4], then

$$\bar{B}_n''(f, x) = n(n-1) \sum_{k=0}^{n-2} \vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{k}{n}) p_{n-2,k}(x).$$

We have

$$\begin{aligned} |\bar{w}(x)\bar{B}_n''(f, x)| &\leq Cn^2\bar{w}(x) \sum_{k=0}^{n-2} |\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{k}{n})| p_{n-2,k}(x) \\ &= Cn^2\bar{w}(x) \left[\sum_{k/n \in A} p_{n-2,k}(x) |\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{k}{n})| + \sum_{x_2 \leq k/n \leq x_3} p_{n-2,k}(x) |\vec{\Delta}_{\frac{1}{n}}^2 P(\frac{k}{n})| \right]. \end{aligned}$$

We can deal with it in accordance with Case 1, and prove it immediately, then the theorem is done. \square

4.2 Proof of Theorem 2

(1) We prove the first inequality of Theorem 2.

Case 1. If $0 \leq \varphi(x) \leq \frac{1}{\sqrt{n}}$, by (2.2), we have

$$|\bar{w}(x)\phi^2(x)\bar{B}_n''(f, x)| = \varphi^2(x) \cdot \frac{\phi^2(x)}{\varphi^2(x)} |\bar{w}(x)\bar{B}_n''(f, x)| \leq Cn\|\bar{w}f\|.$$

Case 2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, by [4], we have

$$\begin{aligned} \bar{B}_n''(f, x) = B_n''(\bar{F}_n, x) &= (\varphi^2(x))^{-1} \sum_{i=0}^2 Q_i(x, n) n^i \sum_{k=0}^n (x - \frac{k}{n})^i \bar{F}_n(\frac{k}{n}) p_{n,k}(x), \\ (\varphi^2(x))^{-1} Q_i(x, n) n^i &\leq C(n/\varphi^2(x))^{1+i/2}. \end{aligned}$$

So

$$\begin{aligned} &|\bar{w}(x)\phi^2(x)\bar{B}_n''(f, x)| \\ &\leq C\bar{w}(x)\phi^2(x) \sum_{i=0}^2 (\frac{n}{\varphi^2(x)})^{1+i/2} \sum_{k=0}^n |x - \frac{k}{n}|^i |\bar{F}_n(\frac{k}{n})| p_{n,k}(x) \\ &= C\bar{w}(x)\phi^2(x) \sum_{i=0}^2 (\frac{n}{\varphi^2(x)})^{1+i/2} \sum_{k/n \in A} |x - \frac{k}{n}|^i |\bar{F}_n(\frac{k}{n})| p_{n,k}(x) \\ &+ C\bar{w}(x)\phi^2(x) \sum_{i=0}^2 (\frac{n}{\varphi^2(x)})^{1+i/2} \sum_{x_2 \leq k/n \leq x_3} |x - \frac{k}{n}|^i |P(\frac{k}{n})| p_{n,k}(x) \\ &:= \sigma_1 + \sigma_2. \end{aligned}$$

Where $A := [0, x_2] \cup [x_3, 1]$. Working as in the proof of Theorem 1, We can get $\sigma_1 \leq Cn\|\bar{w}f\|$, $\sigma_2 \leq Cn\|\bar{w}f\|$. By bringing these facts together, we can immediately get the first inequality of Theorem 2.

(2) If $f \in W_\phi^2$, by (2.1), then

$$\begin{aligned} |\bar{w}(x)\phi^2(x)\bar{B}_n''(f, x)| &\leq n^2 \bar{w}(x)\phi^2(x) \sum_{k=0}^{n-2} |\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{k}{n})| p_{n-2,k}(x) \\ &= n^2 \bar{w}(x)\phi^2(x) \sum_{k=1}^{n-3} |\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{k}{n})| p_{n-2,k}(x) \\ &+ n^2 \bar{w}(x)\phi^2(x) |\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(0)| p_{n-2,0}(x) \\ &+ n^2 \bar{w}(x)\phi^2(x) |\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{n-2}{n})| p_{n-2,n-2}(x) \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{4.5}$$

By [4], if $0 < k < n-2$, we have

$$|\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{k}{n})| \leq Cn^{-1} \int_0^{\frac{2}{n}} |\bar{F}_n''(\frac{k}{n} + u)| du, \tag{4.6}$$

If $k = 0$, we have

$$|\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(0)| \leq C \int_0^{\frac{2}{n}} u |\bar{F}_n''(u)| du, \tag{4.7}$$

Similarly

$$|\vec{\Delta}_{\frac{1}{n}}^2 \bar{F}_n(\frac{n-2}{n})| \leq Cn^{-1} \int_{1-\frac{2}{n}}^1 (1-u) |\bar{F}_n''(u)| du. \quad (4.8)$$

By (4.6), then

$$\begin{aligned} I_1 &\leq Cn\bar{w}(x)\phi^2(x) \sum_{k=1}^{n-3} \int_0^{\frac{2}{n}} |\bar{F}_n''(\frac{k}{n}+u)| dup_{n-2,k}(x) \\ &= Cn\bar{w}(x)\phi^2(x) \sum_{k/n \in A} \int_0^{\frac{2}{n}} |\bar{F}_n''(\frac{k}{n}+u)| dup_{n-2,k}(x) \\ &\quad + Cn\bar{w}(x)\phi^2(x) \sum_{x_2 \leq k/n \leq x_3} \int_0^{\frac{2}{n}} |P''(\frac{k}{n}+u)| dup_{n-2,k}(x) \\ &:= T_1 + T_2. \end{aligned}$$

Where $A := [0, x_2] \cup [x_3, 1]$, P is a linear function. If $\frac{k}{n} \in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n})} \leq C(1+n^{-\frac{\alpha}{2}}|k-nx|^\alpha)$, we have $|k-n\xi| \geq \frac{\sqrt{n}}{2}$, by (3.1), (3.4) and (3.7), then

$$\begin{aligned} T_1 &\leq C\bar{w}(x)\phi^2(x) \|\bar{w}\phi^2 \bar{F}_n''\| \sum_{k/n \in A} p_{n-2,k}(x) \bar{w}^{-1}(\frac{k}{n}) \phi^{-2}(\frac{k}{n}) \\ &\leq C\phi^2(x) \|\bar{w}\phi^2 \bar{F}_n''\| \sum_{k=0}^{n-2} p_{n-2,k}(x) (1+n^{-\frac{\alpha}{2}}|k-nx|^\alpha) \phi^{-2}(\frac{k}{n}) \\ &\leq C\|\bar{w}\phi^2 \bar{F}_n''\| \\ &\leq C\|\bar{w}\phi^2 f''\|. \end{aligned}$$

Working as the Theorem 1, we can get

$$T_2 \leq C\|\bar{w}\phi^2 f''\|.$$

So, we can get

$$I_1 \leq C\|\bar{w}\phi^2 f''\|. \quad (4.9)$$

By (3.7) and (4.7), we have

$$\begin{aligned} I_2 &\leq Cn^2\bar{w}(x)\phi^2(x)(1-x)^{n-2} \int_0^{\frac{2}{n}} u |\bar{F}_n''(u)| du \\ &\leq Cn^2\bar{w}(x)\phi^2(x)(1-x)^{n-2} \|\bar{w}\phi^2 \bar{F}_n''\| \int_0^{\frac{2}{n}} u \bar{w}^{-1}(u) \phi^{-2}(u) du \\ &\leq C\|\bar{w}\phi^2 \bar{F}_n''\| \\ &\leq C\|\bar{w}\phi^2 f''\|. \end{aligned} \quad (4.10)$$

Similarly

$$I_3 \leq C\|\bar{w}\phi^2 f''\|. \quad (4.11)$$

By bringing (4.5), (4.9)-(4.11) together, we can get the second inequality of Theorem 2. \square

Corollary For any $\alpha > 0$, $0 \leq \lambda \leq 1$, we have

$$|\bar{w}(x)\varphi^{2\lambda}(x)\bar{B}_n''(f, x)| \leq \begin{cases} Cn\{\max\{n^{1-\lambda}, \varphi^{2(\lambda-1)}\}\}\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C\|\bar{w}\varphi^{2\lambda}f''\|, & f \in W_{\bar{w}, \lambda}^2. \end{cases} \quad (4.12)$$

4.3 Proof of Theorem 3

4.3.1 The direct theorem

We know

$$\bar{F}_n(t) = \bar{F}_n(x) + \bar{F}'_n(t)(t-x) + \int_x^t (t-u)\bar{F}''_n(u)du, \quad (4.13)$$

$$B_n(t-x, x) = 0. \quad (4.14)$$

According to the definition of W_ϕ^2 , by (4.13) and (4.14), for any $g \in W_\phi^2$, we have $\bar{B}_n(g, x) = B_n(\bar{G}_n(g), x)$, then

$$\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n, x)| = \bar{w}(x)|B_n(R_2(\bar{G}_n, t, x), x)|, \quad (4.15)$$

thereof $R_2(\bar{G}_n, t, x) = \int_x^t (t-u)\bar{G}''_n(u)du$.

$$\begin{aligned} \bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n, x)| &\leq C\bar{w}(x) \sum_{k=1}^{n-1} p_{n,k}(x) \int_x^{\frac{k}{n}} |\frac{k}{n} - u| |\bar{G}''_n(u)| du \\ &\quad + C\bar{w}(x) p_{n,0}(x) \int_0^x u |\bar{G}''_n(u)| du \\ &\quad + C\bar{w}(x) p_{n,n}(x) \int_x^1 (1-u) |\bar{G}''_n(u)| du \\ &:= I_1 + I_2 + I_3. \end{aligned} \quad (4.16)$$

If u between $\frac{k}{n}$ and x , we have

$$\frac{|\frac{k}{n} - u|}{\bar{w}^2(u)} \leq \frac{|\frac{k}{n} - x|}{\bar{w}^2(x)}, \quad \frac{|\frac{k}{n} - u|}{\phi^4(u)} \leq \frac{|\frac{k}{n} - x|}{\phi^4(x)}. \quad (4.17)$$

By (3.4) and (4.17), then

$$\begin{aligned} I_1 &\leq C\|\bar{w}\phi^2\bar{G}''_n\|\bar{w}(x) \sum_{k=1}^{n-1} p_{n,k}(x) \int_x^{\frac{k}{n}} \frac{|\frac{k}{n} - u|}{\bar{w}(u)\phi^2(u)} du \\ &\leq C\|\bar{w}\phi^2\bar{G}''_n\|\bar{w}(x) \sum_{k=1}^{n-1} p_{n,k}(x) \left(\int_x^{\frac{k}{n}} \frac{|\frac{k}{n} - u|}{\bar{w}^2(u)} du \right)^{\frac{1}{2}} \left(\int_x^{\frac{k}{n}} \frac{|\frac{k}{n} - u|}{\phi^4(u)} du \right)^{\frac{1}{2}} \\ &\leq Cn^{-2}\|\bar{w}\phi^2\bar{G}''_n\|\phi^{-2}(x) \sum_{k=0}^{n-1} p_{n,k}(x)(k-nx)^2 \\ &\leq Cn^{-1} \frac{\varphi^2(x)}{\phi^2(x)} \|\bar{w}\phi^2\bar{G}''_n\| \\ &\leq Cn^{-1} \frac{\delta_n^2(x)}{\phi^2(x)} \|\bar{w}\phi^2\bar{G}''_n\| \\ &= C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2\bar{G}''_n\|. \end{aligned} \quad (4.18)$$

For I_2 , when u between $\frac{k}{n}$ and x , we let $k=0$, then $\frac{u}{\bar{w}(u)} \leq \frac{x}{\bar{w}(x)}$, and

$$\begin{aligned} I_2 &\leq C\|\bar{w}\phi^2\bar{G}''_n\|\bar{w}(x) p_{n,0}(x) \int_0^x u \bar{w}^{-1}(u) \phi^{-2}(u) du \\ &\leq C(nx)(1-x)^{n-1} \cdot n^{-1} \frac{\varphi^2(x)}{\phi^2(x)} \|\bar{w}\phi^2\bar{G}''_n\| \\ &\leq C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2\bar{G}''_n\|. \end{aligned} \quad (4.19)$$

Similarly, we have

$$I_3 \leq C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 \bar{G}_n''\|. \quad (4.20)$$

By bringing (4.18)-(4.20), we have

$$\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n, x)| \leq C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 \bar{G}_n''\|. \quad (4.21)$$

By (3.6) and (4.21), when $g \in W_\phi^2$, then

$$\begin{aligned} \bar{w}(x)|g(x) - \bar{B}_n(g, x)| &\leq \bar{w}(x)|g(x) - \bar{G}_n(g, x)| + \bar{w}(x)|\bar{G}_n(g, x) - \bar{B}_n(g, x)| \\ &\leq \bar{w}(x)|g(x) - P(g, x)|_{[x_1, x_4]} + C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 \bar{G}_n''\| \\ &\leq C \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 g''\|. \end{aligned} \quad (4.22)$$

For $f \in C_{\bar{w}}$, we choose proper $g \in W_\phi^2$, by (3.2) and (4.22), then

$$\begin{aligned} \bar{w}(x)|f(x) - \bar{B}_n(f, x)| &\leq \bar{w}(x)|f(x) - g(x)| + \bar{w}(x)|\bar{B}_n(f - g, x)| + \bar{w}(x)|g(x) - \bar{B}_n(g, x)| \\ &\leq C(\|\bar{w}(f - g)\| + \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)} \right)^2 \|\bar{w}\phi^2 g''\|) \\ &\leq C\omega_\phi^2(f, \frac{\delta_n(x)}{\sqrt{n}\phi(x)})_{\bar{w}}. \square \end{aligned}$$

4.3.2 The inverse theorem

The main-part K-functional is given by

$$K_{2,\phi}(f, t^2)_{\bar{w}} = \sup_{0 < h \leq t} \inf_g \{ \|\bar{w}(f - g)\| + t^2 \|\bar{w}\phi^2 g''\|, \quad g' \in A.C_{loc} \}.$$

By [4], we have

$$C^{-1}K_{2,\phi}(f, t^2)_{\bar{w}} \leq \omega_\phi^2(f, t)_{\bar{w}} \leq CK_{2,\phi}(f, t^2)_{\bar{w}}. \quad (4.23)$$

Proof. Let $\delta > 0$, by (4.23), we choose proper g so that

$$\|\bar{w}(f - g)\| \leq C\omega_\phi^2(f, \delta)_{\bar{w}}, \quad \|\bar{w}\phi^2 g''\| \leq C\delta^{-2}\omega_\phi^2(f, \delta)_{\bar{w}}. \quad (4.24)$$

then

$$\begin{aligned} |\bar{w}(x)\Delta_{h\phi}^2 f(x)| &\leq |\bar{w}(x)\Delta_{h\phi}^2(f(x) - \bar{B}_n(f, x))| + |\bar{w}(x)\Delta_{h\phi}^2 \bar{B}_n(f - g, x)| \\ &\quad + |\bar{w}(x)\Delta_{h\phi}^2 \bar{B}_n(g, x)| \\ &\leq \sum_{j=0}^2 C_2^j (n^{-\frac{1}{2}} \frac{\delta_n(x + (1-j)h\phi(x))}{\phi(x + (1-j)h\phi(x))})^{\alpha_0} \\ &\quad + \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \bar{w}(x) \bar{B}_n''(f - g, x + \sum_{k=1}^2 u_k) du_1 du_2 \\ &\quad + \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \bar{w}(x) \bar{B}_n''(g, x + \sum_{k=1}^2 u_k) du_1 du_2 \\ &:= J_1 + J_2 + J_3. \end{aligned} \quad (4.25)$$

Obviously

$$J_1 \leq C((n^{-\frac{1}{2}}\phi^{-1}(x)\delta_n(x))^{\alpha_0}). \quad (4.26)$$

By (2.2) and (4.24), we have

$$\begin{aligned} J_2 &\leq Cn^2\|\bar{w}(f-g)\| \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} du_1 du_2 \\ &\leq Cn^2 h^2 \phi^2(x) \|\bar{w}(f-g)\| \\ &\leq Cn^2 h^2 \phi^2(x) \omega_\phi^2(f, \delta)_{\bar{w}}. \end{aligned} \quad (4.27)$$

By the second inequality of (4.12) and (4.24), we have

$$\begin{aligned} J_2 &\leq Cn\|\bar{w}(f-g)\| \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \varphi^{-2}(x + \sum_{k=1}^2 u_k) du_1 du_2 \\ &\leq Cn h^2 \phi^2(x) \varphi^{-2}(x) \|\bar{w}(f-g)\| \\ &\leq Cn h^2 \phi^2(x) \varphi^{-2}(x) \omega_\phi^2(f, \delta)_{\bar{w}}. \end{aligned} \quad (4.28)$$

By the second inequality of (2.3), (3.3) and (4.24), we have

$$\begin{aligned} J_3 &\leq C\|\bar{w}\phi^2 g''\| \bar{w}(x) \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \bar{w}^{-1}(x + \sum_{k=1}^2 u_k) \phi^{-2}(x + \sum_{k=1}^2 u_k) du_1 du_2 \\ &\leq Ch^2 \|\bar{w}\phi^2 g''\| \\ &\leq Ch^2 \delta^{-2} \omega_\phi^2(f, \delta)_{\bar{w}}. \end{aligned} \quad (4.29)$$

Now, by (4.25)-(4.29), there exists a constant $M > 0$ so that

$$\begin{aligned} |\bar{w}(x)\Delta_{h\phi}^2 f(x)| &\leq C((n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{\alpha_0} \\ &\quad + \min\{n\frac{\phi^2(x)}{\varphi^2(x)}, n^2\phi^2(x)\} h^2 \omega_\phi^2(f, \delta)_{\bar{w}} + h^2 \delta^{-2} \omega_\phi^2(f, \delta)_{\bar{w}}) \\ &\leq C((n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{\alpha_0} \\ &\quad + h^2 M^2 (n^{-\frac{1}{2}}\frac{\varphi(x)}{\phi(x)} + n^{-\frac{1}{2}}\frac{n^{-1/2}}{\phi(x)})^{-2} \omega_\phi^2(f, \delta)_{\bar{w}} + h^2 \delta^{-2} \omega_\phi^2(f, \delta)_{\bar{w}}) \\ &\leq C((n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{\alpha_0} \\ &\quad + h^2 M^2 (n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{-2} \omega_\phi^2(f, \delta)_{\bar{w}} + h^2 \delta^{-2} \omega_\phi^2(f, \delta)_{\bar{w}}). \end{aligned}$$

When $n \geq 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x) < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x) \leq \sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper $x, \delta, n \in N$, so that

$$n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)} \leq \delta < (n-1)^{-\frac{1}{2}}\frac{\delta_{n-1}(x)}{\phi(x)},$$

Therefore

$$|\bar{w}(x)\Delta_{h\phi}^2 f(x)| \leq C\{\delta^{\alpha_0} + h^2\delta^{-2}\omega_\phi^2(f, \delta)_{\bar{w}}\}.$$

Which implies

$$\omega_\phi^2(f, t)_{\bar{w}} \leq C\{\delta^{\alpha_0} + h^2\delta^{-2}\omega_\phi^2(f, \delta)_{\bar{w}}\}.$$

So, by Berens-Lorentz lemma in [4], we get

$$\omega_\phi^2(f, t)_{\bar{w}} \leq Ct^{\alpha_0}. \square$$

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